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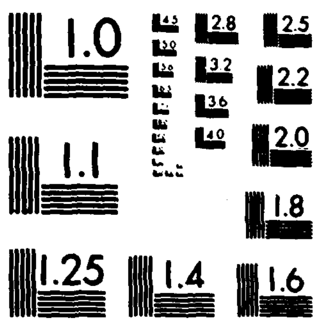

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ON THE CONSTRUCTION OF BIB DESIGNS WITH VARIABLE SUPPORT SIZES

by
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Department of Mathematics
University of Illinois at Chicago

October, 1980

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ON THE CONSTRUCTION OF BIB DESIGNS WITH VARIABLE SUPPORT SIZES

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ABSTRACT

A balanced incomplete block (BIB) design with b blocks is said to have support size b^* when exactly b^* of the b blocks are distinct. The importance and the applications of BIB designs with $b^* < b$ in design of experiments and controlled sampling were explained in detail in Foody and Hedayat (1977) and Wynn (1977). A method of constructing BIB designs with various support sizes from known designs is introduced. This method, together with another method called, "trade-off", which was introduced by Hedayat and Li (1979) are utilized to construct BIB designs with $v = 8$, $k = 4$ in particular. A table of BIB designs with $14 \leq b^* \leq 70$, $b^* \neq 15,16,17,19$ with $v = 8$ and $k = 4$ is included.

1. Introduction: Following the standard notation we consider BIB designs with parameters v,b,r,k and λ. It is known that BIB designs with repeated blocks are useful in experimental designs and in controlled sampling (see Foody and Hedayat (1977) and Wynn (1977) for detail). The structure of BIB designs with repeated blocks has interested researchers since the 60's, for example, see Parker (1963), Seiden (1963), Stanton and Sprott (1964), Mann (1969), van Lint and Ryser (1972), van Lint (1973,1974) and Wynn (1977). More recently Foody and Hedayat (1977), Hedayat and Li (1979) and Hedayat and Khosrovshahi (1979) systematically studied the techniques of constructing BIB designs with various support sizes for a given v and k. Foody and Hedayat (1977) showed that the combinatorial problem of searching for BIB designs with repeated blocks is equivalent to the algebraic problem of finding solutions to a set of homogeneous linear equations. A table of designs based on $v = 8$ and $k = 3$ with $22 \leq b^* \leq 56$ were produced by using this equivalence. Hedayat and Li (1979) introduced a method called "trade off" which was utilized to construct BIB designs with $v = 7$ and $k = 3$ with all possible support sizes. Hedayat and Khosrovshahi (1979) utilized a linear algebraic technique to study BIB designs with $v = 6$ and $k = 3$ and produced a corresponding table of designs on all possible support sizes. It can be found that these tables provided by the above authors contain only the designs with minimum b corresponding to each support size b^*.

This report is a continuation of the above research. In Section 3, we introduce a method of constructing BIB designs with various support sizes from known designs. Using this method we have produced most of the designs listed in Table 1 which is based on $v = 8$ and $k = 4$, except for $b^* = 23, 41, 42, 55$. However this method is applicable only when some known designs exist and satisfy certain conditions. In Section 4, we study the "trade off" method and give a more explicit way to use the "trades". The designs corresponding to $b^* = 23, 41, 42, 55$ in Table 1 are constructed through this method. In the Appendix we have proved that there is no (v,k) trade of volume t for $t = 1,2,3$. The nonexistence of a (v,k) trade of volume 5, for $k \leq 4$ is proved by computation. It is important to note that for a given number b^*, if d_1 is a $BIB(v,b_1,r_1,k,\lambda_1|b^*)$, such that b_1 is minimum among those designs with support size b^* and if d_1 contains a BIB design d_2 which contains b_{min} blocks such that b_{min} is the minimum positive integer solution for b satisfying (i) $bk = vr$ and (ii) $\lambda(v-1) = r(k-1)$ with λ, r positive integers, then the existence of a $BIB(v,b,r,k,\lambda|b^*)$ is always guaranteed for $b \geq b_1$ and b satisfying (i) and (ii). The designs listed in Table 1 are selected to fit the above properties. Hereafter, for simplicity, we shall refer to the $BIB(8,14,7,4,3)$ presented in the first column of Table 1 as the first design. Except for $b^* = 23$, all the BIB designs found in Table 1 contain the first design. While the design for $b = 28$, $b^* = 23$ in Table 1 is actually a combination of two $BIB(8,14,7,4,3)$ (see Example 4.1).

2. Definitions and notation.

Let $V = \{1,2,\ldots,v\}$ and let $v\Sigma k$ be the set of all distinct subsets of size k based on V. Elements of $v\Sigma k$ will be called blocks. A block of size 2 will be referred to as a pair. A block of size k consisting of elements $x_1,x_2,\ldots,x_k$ will either be denoted by $(x_1\ x_2\ \ldots\ x_k)$ or $x_1x_2\ \ldots\ x_k$, while the order among the k elements are immaterial.

A balanced incomplete block design, d, with the parameters v,b,r,k and λ, written BIB(v,b,r,k,λ), is a collection of b elements of $v\Sigma k$ with the properties that:

(i) each element of V occurs in exactly r blocks;

(ii) each element of $v\Sigma 2$ appears together in exactly λ blocks.

Note that repeated blocks are allowed in a BIB design. The number of distinct blocks in a BIB design d, denoted by b^*, is called the support size of d and the support of d is defined to be the collection of b^* distinct blocks contained in d.

We will denote a BIB(v,b,r,k,λ) with support size b^* by BIB$(v,b,r,k,\lambda \mid b^*)$. A BIB design with $b = b^* = \binom{v}{k}$ (the cardinality of $v\Sigma k$) is denoted by $d(v,k)$ and referred to as the trivial BIB design based on v and k.

Order the blocks in $v\Sigma k$ and let B_i be the ith element of $v\Sigma k$, we identify B_i with the $\binom{v}{k}$-dimensional column vector whose entries are zeros except that the ith entry is one. Then a balanced incomplete block design can also be identified with a

$\binom{v}{k}$-dimensional column vector $F = (f_1, f_2, \ldots,)'$, in which f_i denotes the frequency of the ith element of $v\Sigma k$ in the design. In terms of this, a $BIB(v,b,r,k,\lambda \mid b^*)$ can be regarded as $\sum_i f_i B_i$ with f_i nonnegative integers and $B_i \in v\Sigma k$ such that:

(i) $\sum_i f_i = b$ and $\sum_{f_i \neq 0} 1 = b^*$.

(ii) $\sum_{B_i \ni x} f_i = r$ and $\sum_{B_i \supset (xy)} f_i = \lambda$ for any $x \in V$ and $(xy) \in v\Sigma 2$.

Throughout this report, $\underset{\sim}{0}_{mxn}$ will denote an mxn zero matrix and J_{mxn} is an mxn matrix with all entries equal to one. To avoid messy expressions the dimensions of matrices should be deduced from the context if they are not explicitly specified.

3. A Method of constructing BIB designs.

Let S_{k1} and S_{k2} be subsets of $v\Sigma k$ such that $S_{k1} = \{x_1 \ldots x_k;\ x_i \neq v,\ \text{for all}\ i = 1,2,\ldots,v\}$ and $S_{k2} = \{x_1 \ldots x_k;\ x_i = v\ \text{for some}\ i\}$. Clearly, $S_{k1} \cap S_{k2} = \phi$ and $v\Sigma k = S_{k1} \cup S_{k2}$. Assume the blocks in $v\Sigma k$ are ordered in the following manner: the blocks in S_{k1} precede those in S_{k2} and for each i, the blocks in S_{ki} are ordered lexicographically. Let P be the incidence matrix of pairs versus blocks based on V. i.e., P is a $\binom{v}{2}$ by $\binom{v}{k}$ matrix with $p_{ij} = 1$ if the ith pair is contained in the jth block and $p_{ij} = 0$ otherwise. Then

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$

where P_{ij} is the incidence matrix of the pairs in S_{2i} versus the blocks in S_{kj}, $i = 1,2$, $j = 1,2$. Indeed $P_{21} = \underset{\sim}{0}$ is a zero matrix.

<u>Theorem 3.1</u>. Suppose d_i is a BIB$(v-1, b_i, r_i, k_i, \lambda_i | b_i^*)$, $i = 1,2$ and $k_1 = k_2 + 1 = k$. Then there exists a BIB$(v, b_1+b_2, r_1+r_2, k, \lambda_1+\lambda_2 | b_1^*+b_2^*)$ if and only if $r_2 = \lambda_1+\lambda_2$.

<u>Proof</u>: Let d_3 be the new design obtained from d_2 by augmenting the vth element to each block of d_2. Note that $d_1 \cap d_3 = \phi$ and d_3 is not a BIB design. Let $d = d_1 \cup d_3$ and let $F_i = (f_{i1}, f_{i2}, \ldots)'$ be the $\binom{v}{k}$-dimensional column vector in which f_{ij} is the frequency of the j^{th} element of $v\Gamma k$ in d_i, $i = 1,3$. Then $F = F_1 + F_3$ is the frequency vector associated with d based on P. According to the way we ordered $v\Gamma k$, $F_1 = [\bar{F}_1' \vdots \underset{\sim}{0}]'$ and $F_3 = [\underset{\sim}{0} \vdots \bar{F}_3']'$, where $\bar{F}_1$ is a $\binom{v-1}{k}$-dimensional column vector and $\bar{F}_3$ is a $\binom{v-1}{k-1}$-dimensional column vector. Then

$$PF = P(F_1+F_3) = \left[\begin{array}{cc} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] \left[\begin{array}{c} \bar{F}_1 \\ \hline \bar{F}_3 \end{array}\right]$$

$$= \left[\begin{array}{c} P_{11}\bar{F}_1 + P_{12}\bar{F}_3 \\ \hline P_{21}\bar{F}_1 + P_{22}\bar{F}_3 \end{array}\right] = \left[\begin{array}{c} \lambda_1 + \lambda_2 \\ \hline r_2 \end{array}\right] .$$

Thus d is a BIB design if and only if $\lambda_1 + \lambda_2 = r_2$.

Example 3.1. Let d be BIB(8,14,7,4,3|14), the first design in Table 1. Let d_1 be consisting of five copies of d and $d_2 = d(8,3)$, we have:

$$r_1 = 35,\ k_1 = 4,\ \lambda_1 = 15,\ b_1 = 70,\ b_1^* = 14;$$

$$r_2 = 21,\ k_2 = 3,\ \lambda_2 = 6,\ b_2 = 56,\ b_2^* = 56$$

and $$r_2 = \lambda_1 + \lambda_2 = 21.$$

Augment 9 to each block of d_2 to obtain d_3. Then $d_1 \cup d_3$ is a BIB(9,126,56,4,21|70).

Corollary 3.1. (i) Suppose $v = 2k$ and k is even. Let d_1 be BIB($(2k-1)$, $(2k-1)t_1$, kt_1, k, $kt_{1/2}$ | b_1^*) and d_2 be BIB($(2k-1)$, $(2k-1)t_2$, $(k-1)t_2$, $k-1$, $(k-2)t_{2/2}$ | b_2^*). Then there exists a BIB($2k$, $2(2k-1)t$, $(2k-1)t$, k, $(k-1)t$ | $b_1^* + b_2^*$) if and only if $t_1 = t_2 = t$.

(ii) Suppose $v = 2k$ and k is odd. Let d_1 be BIB($(2k-1)$, $2(2k-1)t_1$, $2kt_1$, k, $kt_1 \mid b_1^*$) and d_2 be BIB($(2k-1)$, $2(2k-1)t_2$, $2(2k-1)t_2$, $k-1$, $(k-2)t_2 \mid b_2^*$). Then there exists a BIB($2k$, $4(2k-1)t$, $2(2k-1)t$, k, $2(k-1)t \mid b_1^* + b_2^*$) if and only if $t_1 = t_2 = t$.

Since a BIB($(2k-1)$, b, r, k, λ) is indeed the complement of a BIB($(2k-1)$, b, $b-r$, $v-k$, $b-2r+\lambda$), an alternative way to describe the above is that suppose d_i is BIB($(2k-1)$, b, r, $k-1$, $\lambda \mid b_i^*$), $i = 1,2$, then $d_1' \cup d_2'$ is a BIB($2k$, $2b$, b, k, $b-2r+2\lambda \mid b_1^* + b_2^*$) where d_1' consists of the blocks obtained by augmenting the $(2k)$th element to each block of d_1 and d_2' is the complementary design of d_2 based on $(2k-1)$ elements.

<u>Remark 3.1</u>. When $d_1 = d_2$ then $d_1' \cup d_2'$ is indeed self complementary and hence a 3-design. (See Hedayat and John (1974).)

<u>Example 3.2</u>. Let $v = 8$ and $k = 4$.

Let $d_1 = \{124, 235, 346, 457, 561, 672, 713\}$ and
$d_2 = \{124, 135, 167, 237, 256, 346, 457\}$.

Then $d_1' = \{1248, 2358, 3468, 4578, 5618, 6728, 7138\}$ and
$d_2' = \{3567, 2467, 2345, 1456, 1347, 1257, 1236\}$.

It can be easily checked that $d_1' \cup d_2'$ is a BIB($8,14,7,4,3 \mid 14$).

Example 3.3. Let $v = 8$ and $k = 4$.

Let $d_1 = d_2 = \{124, 235, 346, 457, 561, 672, 713\}$.

Then $d_1' = \{1248, 2358, 3468, 4578, 5618, 6728, 7138\}$,

$d_2' = \{3567, 1467, 1257, 1236, 2347, 1345, 2456\}$

and $d_1' \cup d_2'$ is not only a BIB(8,14,7,4,3|14) but also a 3-design.

Hedayat and Li (1979) have produced a table of designs based on $v = 7$ and $k = 3$ with all possible support sizes. We can now use their table and the above method to construct BIB designs based on $v = 8$ and $k = 4$. Most designs except for $b^* = 23, 41, 42, 55$ in Table 1 are found this way.

4. The trade off method.

If B_i is a block in $v\Sigma k$ and t_i is an integer, the collection

$$\mathfrak{T} = \left\{\sum_i t_i B_i : \sum_{i:B_i \supset (xy)} t_i = 0 \quad \text{for all} \quad (xy) \in v\Sigma 2\right\}$$

is of particular interest. Following Hedayat and Li (1979), elements of $\mathfrak{T}$ are called (v,k) trades. The sum of positive t_i's in a (v,k) trade is referred to as the volume of the trade. (It can be easily seen that the sums of positive t_i's and negative t_i's in a (v,k) trade are equal). Whenever a BIB(v,b,r,k,λ) exists, any other design with the same parameters can be obtained by adding proper elements of $\mathfrak{T}$.

<u>Example 4.1</u>. Let $v = 8$ and $k = 4$.

Then (1238) + (1456) + (1478) + (1678) + (2467) + (2578) + (3458) + (3468) + (3567) - (1278) - (1358) - (1467) - (1468) - (2348) - (2567) - (3456) - (3678) - (4578) represents a trade of volume 9. When this trade is added to the BIB design d_1 = (1236) + (1245) + (1278) + (1357) + (1358) + (1467) + (1468) + (2347) + (2348) + (2567) + (2568) + (3456) + (3678) + (4578), we obtain another BIB design d_2 = (1236) + (1238) + (1245) + (1357) + (1456) + (1478) + (1678) + (2347) + (2467) + (2568) + (2578) + (3458) + (3468) + (3567). In other words, from d_1 the nine blocks (1278), (1358), (1467), (1468), (2348), (2567), (3456), (3678) and (4578) have been traded for the blocks (1238), (1456), (1478), (1678), (2467), (2578), (3458), (3468) and (3567) to obtain the second design d_2.

The design $d_1 + d_2$ being a BIB(8, 28, 14, 4, 6|23) is listed in Table 1.

It is obvious that the sum or the difference of any two (v,k) trades is still a (v,k) trade. Therefore $\mathfrak{J}$ forms a Z-module.

Let S_v be the symmetric group on V. Let $T = \sum_i t_i B_i$ be a (v,k) trade. For $\sigma \in S_v$, let $T^\sigma = \sum_i t_i B_i^\sigma$ where $B^\sigma = (\sigma(x_1)\ \sigma(x_2)\ \ldots\ \sigma(x_k))$, if $B = (x_1\ x_2\ \ldots\ x_k)$.

The following theorem gives a generator for the (v,k) trades.

<u>Theorem</u> <u>4.1</u>. (Graham, Li and Li).

The module $\mathcal{M}$ for (v,k) trades is generated over Z by the collection $\{T^{\sigma} : \sigma \in S_v\}$ where

$$\begin{aligned} T = &(135W) + (245W) + (236W) + (146W) \\ &- (246W) - (136W) - (145W) - (235W) \end{aligned} \tag{4.1}$$

and $W = 78 \ldots (k+3)$.

<u>Theorem</u> <u>4.2</u>.

Let $\{B_i : i = 1,2,3,4\}$ be a collection of blocks of size k on V. Suppose $\{B_i\}$ has the following properties:

(i) $\bigcup_{i=1}^{4} B_i = \{x_1, x_2, \ldots, x_6, x_1, \ldots, x_{k+3}\}$ and

$\bigcap_{i=1}^{4} B_i = \{x_7, x_8, \ldots, x_{k+3}\}$,

(ii) each element of $\{x_1, \ldots, x_6\}$ occurs twice in $\bigcup_1^4 B_i$, and each pair of $\{x_1, \ldots, x_6\}$ occurs together either zero or once in $\bigcup_1^4 B_i$.

For each i, let $\overline{B}_i$ be such that $B_i \cup \overline{B}_i = \{x_1, \ldots, x_6, x_7, \ldots, x_{k+3}\}$ and $B_i \cap \overline{B}_i = \{x_7, x_8, \ldots, x_{k+3}\}$. Then $\overline{B}_i$ is a block of size k and $T = \sum_1^4 B_i - \sum_1^4 \overline{B}_i$ is a (v,k) trade of volume 4.

<u>Proof</u>: Let $x_i, x_j \in \{x_1, \ldots, x_6, x_7, \ldots, x_{k+3}\}$, also let $n(x_i, x_j)$ and $\bar{n}(x_i, x_j)$ denote the number of times that x_i and x_j occur together in $\cup B_i$ and $\cup \bar{B}_i$, respectively.

<u>Case i</u>. $x_i, x_j \in \{x_7, \ldots, x_{k+3}\}$, then $n(x_i, x_j) = \bar{n}(x_i, x_j) = 4$.

<u>Case ii</u>. $x_i \in \{x_1, \ldots, x_6\}$ but $x_j \in \{x_7, \ldots, x_{k+3}\}$, then $n(x_i, x_j) = 2$. By (ii) we may assume $x_i \in B_1 \cap B_2$ but $x_i \notin B_3 \cup B_4$. Hence $x_i \notin \bar{B}_1 \cup \bar{B}_2$ and $x_i \in \bar{B}_3 \cap \bar{B}_4$ and $\bar{n}(x_i, x_j) = 2$.

<u>Case iii</u>. $x_i, x_j \in \{x_1, \ldots, x_6\}$, either $n(x_i, x_j) = 0$ or 1.

If $n(x_i, x_j) = 0$, we may assume $x_i \in B_1 \cap B_2$ but $x_i \notin B_3 \cup B_4$ and $x_j \notin B_1 \cup B_2$ but $x_j \in B_3 \cup B_4$. This implies $x_i \notin \bar{B}_1 \cup \bar{B}_2$ but $x_i \in \bar{B}_3 \cap \bar{B}_4$ and $x_j \in \bar{B}_1 \cap \bar{B}_2$ but $x_j \notin \bar{B}_3 \cup \bar{B}_4$ and $\bar{n}(x_i, x_j) = 0$.

If $n(x_i, x_j) = 1$, we may assume $x_i \in B_1 \cap B_2$ and $x_j \in B_2 \cap B_3$ but $x_i \notin B_3 \cup B_4$, $x_j \notin B_1 \cup B_4$. This implies that $x_i \notin \bar{B}_1 \cup \bar{B}_2$, $x_j \notin \bar{B}_2 \cup \bar{B}_3$ but $x_j \in \bar{B}_3 \cap \bar{B}_4$, $x_j \in \bar{B}_1 \cap \bar{B}_4$ and hence $\bar{n}(x_i, x_j) = 1$.

Thus $\bar{n}(x_i, x_j) = n(x_i, x_j)$ for all $x_i, x_j \in \{x_1, \ldots, x_{k+3}\}$ and $T = \sum_1^4 B_i - \sum_1^4 \bar{B}_i$ is a (v,k) trade of volume 4.

Let $\mathcal{C}$ denote the collection of trades of volume 4 with the properties described in Theorem 4.2. We have the following:

<u>Theorem 4.3</u>. $\mathcal{C} = \{T^\sigma; \sigma \in S_v\}$, where T is the same as (4.1). Hence the module $\mathfrak{T}$ for (v,k) trades is generated over Z by $\mathcal{C}$.

Proof: Let $T_1 \in \mathcal{S}$, we may assume $T_1 = \sum_1^4 B_i - \sum_1^4 \overline{B}_i$ with $B_1 = x_1x_2x_3W'$, $B_2 = x_1x_4x_5W'$, $B_3 = x_2x_4x_6W'$, $B_4 = x_3x_5x_6W'$ and $\overline{B}_1 = x_4x_5x_6W'$, $\overline{B}_2 = x_2x_3x_6W'$, $\overline{B}_3 = x_1x_3x_5W'$, $\overline{B}_4 = x_1x_2x_4W'$, $W' = x_7x_8 \ldots x_{k+3}$.

Let $x_1 = \sigma(1)$, $x_2 = \sigma(6)$, $x_3 = \sigma(4)$, $x_4 = \sigma(3)$, $x_5 = \sigma(5)$, $x_6 = \sigma(2)$, and $x_j = \sigma(j)$ for $j = 7,8,\ldots,k+3$. Then $T_1 = T^\sigma$.

On the other hand, let $\sigma \in S_v$, then $T^\sigma = \sum_1^4 B_i^\sigma - \sum \overline{B}_i^\sigma$ where

$$\bigcup_1^4 B_i^\sigma = \{\sigma(1),\sigma(2),\ldots,\sigma(6)\} \cup \sigma(W), \quad \bigcap_1^4 B_i^\sigma = \sigma(W)$$

and

$$B_i^\sigma \cup \overline{B}_i^\sigma = \{\sigma(1),\sigma(2),\ldots,\sigma(6)\} \cup \sigma(W), \quad B_i^\sigma \cap \overline{B}_i^\sigma = \sigma(W).$$

Here $W = \{7,8,\ldots,(k+3)\}$.

Let $x_i = \sigma(i)$, then T^σ has the properties as described in Theorem 4.2, i.e., $T^\sigma \in \mathcal{S}$. Hence we have the desired result.

Although Graham, Li and Li (1980) provided a basis for the (v,k) trades, the collection $\mathcal{S}$ is indeed more practical in constructing new designs from a given BIB design.

Example 4.1. Let $d_{x,y}$ denote the design presented in Table 1 corresponding to $b = x$ and $b^* = y$. Let $T = (1234) + (1468) + (2458) + (3456) - (4568) - (2345) - (1346) - (1248)$, then $T \in \mathcal{S}$. Add T to $d_{42,39}$, we obtain the design $d_{42,41}$. Now let $T = (1245) + (1268) + (1348) + (1356) - (1368) - (1345) - (1256) - (1248)$, which is also in $\mathcal{S}$. Add T to $d_{42,41}$, we obtain the design $d_{42,42}$ in Table 1.

As we have mentioned before, the designs in Table 1 with $b = 28$, $b^* = 23$ and $b = 56$, $b^* = 55$ are similarly constructed by repeated application of trades in $\mathcal{S}$.

Note that for given v and k, if b_{min} is the minimum positive integer solution for b satisfying (i) $bk = vr$ and (ii) $\lambda(v-1) = r(k-1)$ with λ, r positive integers, then any other solution b must be a multiple of b_{min}. If a BIB design d with parameters v, k, b_{min} exists, other BIB designs based on v and k should be able to be constructed from md by adding trades of the form $\sum t_i T_i$ with $t_i \in Z$ and $T_i \in \mathcal{S}$ to md, where md denotes m copies of d, $m = 1,2,\ldots$. It is important to note that without using the table provided by Hedayat and Li (1979), we can still construct BIB designs based on $v = 8$ and $k = 4$ with all possible support sizes as long as we have the first design. As a matter of fact, designs with minimum b corresponding to each b^*, $14 \leq b^* \leq 70$ and $b^* \neq 15,16,17,19$ have also been constructed through the above method without any difficulty.

5. Final remarks.

(i) It is known that $b^* \geq v$ and $b^* \neq v+1$ for any v. Also when $b^* = v$, the designs are uniform (i.e., all the blocks repeat the same number of times.) (See Foody (1980) and van Lint and Ryser (1972).) From Foody (1980), any design with $v = 8$, $k = 4$ $b^* = 10$ if it exists is also uniform. In Foody (1980), Proposition 2.3 leads the nonexistence of designs with $v = 8$, $k = 4$ and $b^* = 8,10$. Therefore when $v = 8$ and $k = 4$, $11 \leq b^* \leq \binom{8}{4}$.

(ii) In this report, we provided a table of designs based on $v = 8$ and $k = 4$ which has minimum b corresponding to each b^* such that $b^* \geq 14$ and $b^* \neq 15,16,17,19$. Whether there exist $BIB(8,14m,7m,4,3m|b^*)$ designs with $b^* \in \{11,12,13,15,16,17,19\}$ is still unknown.

(iii) In the Appendix, we have shown that there is no (v,k) trade of volume 1,2,3 and no (v,k) trade of volume 5 for $k \leq 4$. Hence if any of the above missing designs exists for $b = 28$, it has to be indecomposable.

(iv) Whether there is a k such that a (v,k) trade of volume 5 exists is an interesting unsolved problem.

BIB designs with v = 8 and k = 4

b*	14	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
b	14	28	28	28	28	28	28	28	28	28	28	42	42	42	42	42	42	42	42	42	42	42	42	42	42	56	56	56	56	56	56	56	56	56	56	56	56	56	56	70	70	70	70	70	70	70	70	84	70	84	84	84	70
1234	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	1	1	2	2	1	1	1	1	1	1	1	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1235	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	1	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1236	1	1	2	2	1	2	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	2	1
1237	-	1	-	-	1	-	1	1	1	1	1	1	2	1	2	2	2	2	2	1	1	1	1	1	1	-	-	1	1	-	-	-	-	-	-	-	-	1	-	2	2	1	1	1	1	1	1	1	1	1	1	1	1
1238	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	3	1	1	-	1	-	-	-	-	1	-	1	1	1	1	-	1	-	1	1	1	-	1	1	1	1	1	1
1245	-	-	-	-	-	2	-	1	-	1	1	1	-	-	1	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	1	-	1	1	1	1	1	1	1	1	2	1	1	1	2	1
1246	-	-	-	-	-	-	1	-	1	-	-	-	-	1	-	1	-	1	1	-	-	1	1	1	1	1	1	-	-	1	1	1	1	1	1	1	1	-	1	1	1	-	-	1	-	1	1	-	1	1	1	1	1
1247	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	1	1	1	1	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	2	1	1	2	1	1	1	1	1
1248	1	2	1	1	2	-	2	2	1	1	1	3	3	3	3	3	2	2	2	2	3	3	1	2	1	1	1	1	2	1	2	2	1	1	1	1	1	1	1	2	1	2	1	2	1	2	1	1	1	1	1	2	1
1256	-	1	-	-	1	-	-	-	-	-	-	-	2	1	1	1	1	1	1	1	1	1	1	1	-	1	1	1	1	-	-	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1257	1	1	2	2	1	-	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
1258	-	-	1	-	-	-	-	-	1	1	-	-	-	-	-	-	-	-	1	1	-	-	1	-	-	-	1	1	1	1	1	1	1	1	-	1	-	1	1	1	1	1	1	-	1	1	1	1	1	1	1	1	1
1267	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	1	1	1	1	-	-	-	-	-	-	1	1	1	1	-	-	1	1	1	1	-	-	1	1	1	1	1	1
1268	-	-	-	1	-	-	-	-	-	-	1	-	-	-	-	-	1	1	-	-	-	-	-	-	1	-	-	-	1	-	1	1	1	1	1	1	1	1	-	-	2	-	2	1	2	-	2	2	2	2	2	1	1
1278	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	-	-	1	1	1	1	1	-	1	1	1	1	1	1	-	1	1	1	-	1	1	1	1
1345	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	-	1	1	1	1	1	1	1	-	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1346	-	1	-	-	1	-	-	-	-	-	-	-	1	1	1	-	1	-	-	1	1	1	1	-	-	1	1	-	-	-	-	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1347	1	1	2	2	1	-	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
1348	-	-	1	1	-	-	-	-	1	1	1	-	-	-	-	-	1	1	1	1	-	-	-	-	1	-	-	-	-	-	-	-	1	1	-	1	-	-	-	-	1	-	1	-	1	-	1	1	1	1	2	1	1
1356	-	-	-	-	-	-	1	1	1	1	1	1	-	-	1	1	1	1	1	-	-	-	-	-	1	-	-	1	1	-	-	-	-	-	-	-	-	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1357	-	-	-	-	-	2	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	1	1	1	2	2	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	2	1
1358	1	2	1	1	2	1	2	2	1	1	1	2	2	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	2	1	2	2	1	1	1	1	1	1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
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1368	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	1	-	-	1	1	1	1	1	1	-	-	2	2	1	2	1	1	1	1	1	1	1	1	1	1	2	1	2	1	1	1	2	1	1	1	1	1	1
1378	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	1	-	-	-	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1456	1	1	2	2	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	1	2	1	1	1	1	1	1	1	1	2	2	1	2	1	1	2	1	2	2	2	1
1457	-	1	-	-	1	-	1	-	1	-	-	-	1	-	1	1	1	1	1	1	1	-	-	-	-	-	-	1	1	-	-	-	-	-	-	1	1	1	1	-	-	-	-	1	-	1	1	-	1	1	1	-	1
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1467	-	-	-	-	-	1	-	1	-	1	1	1	1	-	1	1	1	1	1	1	1	-	-	-	-	-	-	1	1	1	1	1	1	1	1	-	-	1	-	2	2	1	1	1	1	1	1	2	1	1	1	2	1
1468	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	1	-	-	1	-	1	1	-	-	-	-	-	1	1	1	-	1	-	-	1	1	-	2	1	2	1	1	1
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1567	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	1	1	1	1	1	-	-	1	1	1	1	1	1	1	1	-	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1
1568	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	1	2	-	-	-	-	-	-	1	1	1	1	1	1	1	1	-	1	-	1	-	1	-	-	-	-	1	1	1
1578	-	-	-	1	-	-	-	-	-	-	1	1	1	-	1	1	1	1	1	1	1	1	-	1	1	1	-	-	1	-	1	1	-	-	-	-	-	1	-	-	1	-	1	-	2	-	1	2	2	2	2	1	1
1678	1	2	2	1	2	1	2	2	2	2	1	2	2	3	2	2	2	2	2	2	1	1	2	1	1	1	2	2	1	2	1	1	1	1	1	1	1	1	1	2	1	2	1	2	1	2	1	1	1	1	1	1	1
2345	1	2	2	2	2	-	2	1	2	1	1	2	2	2	2	2	2	2	2	1	1	2	2	1	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
2346	-	-	-	-	-	-	-	1	-	1	1	1	1	-	1	1	1	1	1	1	1	-	-	-	-	-	-	1	1	-	-	-	-	-	-	-	-	1	-	1	1	2	2	1	2	1	1	2	1	1	1	2	1
2347	-	-	-	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	-	-	-	-	1	1	1	1	1	1	1	1	1	1	-	-	-	-	1	-	1	1	1	1	1	1	1	1
2348	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	1	1	-	1	-	-	1	1	1	1	1	1	1	1	2	1	2	1	1	1	2	2	1	2	1	1	1
2356	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2357	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	-	-	1	1	-	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1
2358	-	-	-	1	-	-	-	-	-	-	1	1	1	-	1	1	1	1	1	1	1	1	-	1	1	-	-	-	1	-	1	1	1	1	1	1	1	1	-	-	2	-	2	1	1	-	2	1	1	1	2	1	1
2367	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	1	1	1	2	2	-	-	1	1	2	1	2	2	1	1	-	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1
2368	-	-	1	-	1	-	1	1	1	1	-	1	1	1	1	1	1	1	1	1	1	1	-	1	1	-	-	-	1	-	1	1	-	-	-	-	-	1	1	1	-	1	-	-	-	1	-	-	-	-	-	1	1
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2456	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	-	-	-	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2457	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	-	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1	2	1
2458	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	2	-	-	1	-	1	1	1	1	1	1	1	-	1	1	-	1	-	1	1	1	-	1	1	1	1	1	1
2467	1	2	2	2	2	1	1	1	1	1	1	2	2	1	2	1	2	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
2468	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	1	2	2	-	2	-	-	1	1	1	1	1	-	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1
2478	-	-	1	1	-	-	-	-	1	1	1	-	-	-	-	-	1	1	1	1	-	-	-	-	-	-	-	-	1	-	1	1	-	-	-	-	-	1	-	-	1	-	1	-	1	-	1	1	1	1	1	1	1
2567	-	-	-	-	-	1	1	1	1	1	1	1	-	-	1	1	1	1	1	-	-	-	-	-	-	-	-	-	-	1	1	-	1	-	-	-	-	1	-	1	1	1	1	1	1	1	1	2	1	1	1	1	1
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2578	-	-	-	-	1	1	1	1	-	-	-	1	1	1	1	1	1	1	-	-	1	1	1	1	1	1	1	1	-	1	-	-	-	-	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2678	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	1	-	-	1	1	1	1	1	1	1	2	-	-	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3456	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	1	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1	-	-	1	-	1	1	1	1	1	1	1	1
3457	-	-	-	-	-	-	-	1	-	1	1	1	-	-	1	-	1	-	-	-	-	-	-	-	-	-	-	1	1	1	1	1	1	1	1	-	-	1	-	2	2	2	2	1	2	1	1	2	1	1	1	2	1
3458	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	1	1	1	1	1	1	1	1	-	-	-	-	-	1	1	1	-	1	-	-	1	1	-	1	1	1	1	1	1
3467	-	-	-	-	-	-	1	-	1	-	-	-	-	1	-	1	-	1	1	-	-	1	1	1	1	1	1	1	1	1	1	-	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3468	1	2	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	2	1	1	2	1	2	2	2	2	2	2	2	1	1	2	1	2	1	3	1	2	1	1	1	1	1	2	1
3478	-	-	-	-	1	-	1	1	-	-	-	2	2	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-	-	1	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3567	1	2	2	2	2	1	1	1	1	1	1	2	3	1	2	2	2	2	2	3	3	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1
3568	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	1	1	1	-	1	-	-	1	1	1	1	1	-	1	1	1	1	1	1	2	1	1	2	2	2	2	1	1
3578	-	-	1	-	-	-	-	-	1	1	-	-	-	-	-	-	-	-	1	1	-	-	1	-	-	1	1	1	-	1	-	-	1	1	1	1	1	-	1	1	1	1	1	1	-	1	1	-	-	-	-	1	1
3678	-	-	-	1	-	1	-	-	-	-	1	-	-	-	-	-	1	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	1	-	1	-	1	2	1	2	2	1	1
4567	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	1	1	1	1	1	-	-	-	-	1	-	1	1	1	1	-	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1
4568	-	-	1	1	1	-	1	1	1	1	1	2	2	1	2	2	2	2	2	2	1	1	1	-	-	-	-	-	1	-	1	1	-	-	-	-	-	1	-	-	2	-	2	-	1	-	2	1	1	1	1	2	1
4578	1	2	1	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	2	1	1	1	2	1	2	1	2	1	2	1	2	1	2	2	1	1
4678	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	-	1	1	-	1	1	-	1	-	-	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5678	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	2	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

APPENDIX

Study of trades of volume t

For convenience, we give an alternative definition of a (v,k) trade:

Definition A.1. A (v,k) trade of volume t, $3 \le k < v$, is a pair $D_1:D_2$ such that each D_i is a set of t blocks in $v\Gamma k$ with the properties that (i) $D_1 \cap D_2 = \phi$ and (ii) each pair of elements in V appears together in blocks of D_2 the same number of times as in D_1.

Given v and k, let D be a set of t blocks in $v\Gamma k$. Note that we do not rule out the possibility that D may contain several copies of a block. Order the blocks in D lexicographically. It is natural to identify D as a $t \times v$ matrix A such that $A' = (a_{ji})$ is the incidence matrix of elements in V versus blocks in D with a_{ji} equal to one if the jth element of V is in the ith block of D and zero otherwise. We shall call A the matrix representation of D.

Lemma A.1. Let $D_1:D_2$ be a pair of t blocks in $v\Gamma k$, $3 \le k < v$ and let A_i be the matrix representation of D_i, $i = 1,2$. Then $D_1:D_2$ is a (v,k) trade of volume t if and only if:

(i) $A_i J_{v\times 1} = k J_{t\times 1}$, $i = 1,2$.

(ii) $R(A_1) \cap R(A_2) = \phi$, where $R(A_i)$ is the set of row vectors of A_i, $i = 1,2$.

(iii) $A_1'A_1 = A_2'A_2$, where A_i' is the transpose of A_i, $i = 1,2$.

Proof: Let $\lambda_{j\ell}^{(1)}$ and $\lambda_{j\ell}^{(2)}$ denote the number of replications of the pair (j,ℓ) in D_1 and D_2 respectively. It suffices to note that $D_1:D_2$ is a trade if and only if $\lambda_{j\ell}^{(1)} = \lambda_{j\ell}^{(2)}$, for all $j,\ell \in V$; while $A_1'A_1 = (\lambda_{j\ell}^{(1)})$ and $A_2'A_2 = (\lambda_{j\ell}^{(2)})$.

Lemma A.2. If $D_1:D_2$ is a (v,k) trade of volume t, then $D_1^c:D_2^c$ is a $(v,v-k)$ trade of volume t, where $D^c = \{B^c;\ B\epsilon D\}$ and $B^c = V-B$, the complement of B with respect to V.

Proof: By Lemma 2.1, $D_1^c:D_2^c$ is a trade if and only if

$$B_1'B_1 = B_2'B_2$$

where B_i is the matrix representation of D_i^c. Since $D_1:D_2$ is a trade, then $A_1'A_1 = A_2'A_2$. Here again A_i is the matrix representation of D_i.

Now $B_i = J - A_i$, thus

$$B_1'B_1 = B_2'B_2 \Leftrightarrow (J-A_1)'\ (J-A_1) = (J-A_2)'\ (J-A_2)$$

$$\Leftrightarrow J'(A_2-A_1) = (A_1'-A_2')J.$$

But

$$J'A_i = \begin{bmatrix} r_1^{(i)} & r_2^{(i)} & \cdots\cdots & r_v^{(i)} \\ r_1^{(i)} & r_2^{(i)} & \cdots\cdots & r_v^{(i)} \\ r_1^{(i)} & & \cdots\cdots & r_v^{(i)} \end{bmatrix}$$

where $r_j^{(i)} = \lambda_{jj}^{(i)}$ is the number of replications of the jth element of V in D_i.

$$A_1'A_1 = A_2'A_2 \Rightarrow r_j^{(1)} = r_j^{(2)} \quad \text{for} \quad j = 1,2,\ldots,v.$$
$$\Rightarrow J'A_1 = J'A_2.$$

Therefore $B_1'B_1 = B_2'B_2$ as desired.

Hereafter, let $B_i^{(1)}$ and $B_i^{(2)}$ denote the ith rows of A_1 and A_2 respectively. Also let $C_j^{(1)}$ and $C_j^{(2)}$ denote the jth columns of A_1 and A_2 respectively.

Lemma A.3. Suppose $D_1:D_2$ is a (v,k) trade of volume t. Let r_j denote the replication number of the jth element of V. Then $r_j = 0$ or $2 \leq r_j \leq t$ and $r_j \neq t-1$.

Proof: Assume $r_\ell = t-1 \ (\neq 0)$ for some $\ell \in V$. Let $A_1 = (a_{ij}^{(1)})$ and $A_2 = (a_{ij}^{(2)})$ be the matrix representations of D_1 and D_2 respectively. Assume $a_{i\ell}^{(1)} = 1$ for $1 \leq i \leq t-1$ and $a_{t\ell}^{(1)} = 0$.

Let ℓ' be any element in V.

Case (i): $a_{t\ell'}^{(1)} = 1$, then $\lambda_{\ell\ell'} = c_{\ell}^{(1)'} c_{\ell'}^{(1)} = r_{\ell'} - 1$. By assumption $A_1'A_1 = A_2'A_2 = (\lambda_{ij})$ with $\lambda_{ii} = r_i$, $i,j \in V$. Hence $c_{\ell}^{(2)'} c_{\ell'}^{(2)} = r_{\ell'} - 1$ implies $a_{i\ell'}^{(2)} = 1$ if $a_{i\ell}^{(2)} = 0$.

Case (ii): $a_{t\ell'}^{(1)} = 0$, then $\lambda_{\ell\ell'} = c_{\ell}^{(1)'} c_{\ell'}^{(1)} = r_{\ell'}$. Which implies $a_{i\ell'}^{(2)} = 0$, if $a_{i\ell}^{(2)} = 0$. Then $a_{t\ell'}^{(1)} = a_{i\ell'}^{(2)}$, $\forall \ell' \in V$. i.e., $B_t^{(1)} = B_i^{(2)}$ which is impossible.

Corollary A.1. There is no (v,k) trade of volume 1, 2, and 3.

Note that if $D_1:D_2$ is a (v,k) trade of volume t and $r_j = t$ for some $j \in V$ then we can delete the jth element of V from each block of D_1 and D_2 and obtain a $(v, k-1)$ trade of volume t. Therefore to study the existence of (v,k) trades of volume t, it is enough to search among those with $r_j \neq t$. Moreover, let n be the number of elements of V appearing in D_1 and D_2, then

$$k+3 \leq n \leq kt/2. \tag{A.1}$$

Since if $n < k+3$ the collection of (n,k) trades is void [Graver and Jurkat (1973)]. $n \leq kt/2$ because each element takes at least two positions out of kt positions in the t blocks. Also, let $E_j = \{i\epsilon V;\ r_i = j\}$, $0 \leq j \leq t$, it is easy to verify that

$$\sum_j |E_j| = n$$

and (A.2)

$$\sum_j j|E_j| = kt.$$

From now on, we study the case when $t = 5$. By (A.1) for any (v,k) trade of volume 5, $n \leq 5k/2$. Hence if $D_1:D_2$ is any (v,k) trade of volume 5, for best convenience, their matrix representations will be $5\times\ell$ matrices with $\ell = [5k/2]$, where $[x]$ denotes the greatest integer $\leq x$.

<u>Theorem A.1</u>. If $D_1:D_2$ is a (v,k) trade of volume 5, then any two blocks in D_i, $i = 1,2$, has at most $k-2$ elements in common.

<u>Proof</u>: It is enough to prove that the assertion is true for $i = 1$. Let A_1 and A_2 be the matrix representations of D_1 and D_2 respectively. Assume that

$$A_1 = \left(\begin{array}{c:c:c} J_{2\times(k-1)} & C & O_{2\times(\ell-k-1)} \\ \hdashline S_{11} & S_{12} & S_{13} \end{array}\right)$$

with $C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ or $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

<u>Case (i)</u>: $r_j = 2$ <u>for some</u> $1 \leq j \leq k-1$. <u>We may assume that</u> $j = 1$. <u>Then</u>

$$S_{11} = \left(\begin{array}{c:c} 0 & \\ 0 & S_{14} \\ 0 & \end{array}\right)$$

and

$$A_1'A_1 = \left(\begin{array}{c|c} J' & S_{11}' \\ \hline C' & S_{12}' \\ \hline \underset{\sim}{0}' & S_{13}' \end{array}\right) \left(\begin{array}{c|c|c} J & C & \underset{\sim}{0} \\ \hline S_{11} & S_{12} & S_{13} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} J'J + S_{11}'S_{11} & J'C + S_{11}'S_{12} & S_{11}'S_{13} \\ \hline C'J + S_{12}'S_{11} & C'C + S_{12}'S_{12} & S_{12}'S_{13} \\ \hline S_{13}'S_{11} & S_{13}'S_{12} & S_{13}'S_{13} \end{array}\right) \qquad (A.3)$$

<u>where</u>

$$J'J + S_{11}'S_{11} = 2J_{(k-1)\times(k-1)} + \left(\begin{array}{c} 000 \\ \hline S_{14}' \end{array}\right) \left(\begin{array}{c|c} 0 & \\ 0 & S_{14} \\ 0 & \end{array}\right)$$

$$= 2J_{(k-1)\times(k-1)} + \left(\begin{array}{c|c} 0 & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & S_{14}'S_{14} \end{array}\right)$$

$$= \left(\begin{array}{c|c} 2 & 2 \ \dots \ 2 \\ \hline 2 & \\ \vdots & 2J_{(k-2)\times(k-2)} + S_{14}'S_{14} \\ 2 & \end{array}\right)$$

<u>implies</u> $\lambda_{ij} = 2$ <u>for</u> $1 \le j \le k - 1.$ (A.4)

(a) *If* $C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, *then*

$$J'C + S_{11}'S_{12} = \begin{pmatrix} 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 \;\cdots\; 0 \\ S_1 \end{pmatrix} = \begin{pmatrix} 2 \;\cdots\; 0 \\ S_2 \end{pmatrix}$$

and $S_{11}'S_{13} = \begin{pmatrix} 0 \;\cdots\; 0 \\ S_3 \end{pmatrix}$, *which implies*

$$\lambda_{1k} = 2 \quad \text{and} \quad \lambda_{ij} = 0 \quad \text{for} \quad k < j \leq v. \tag{A.5}$$

Since $A_1'A_1 = A_2'A_2 = (\lambda_{ij})$, *(A.4) and (A.5) imply that* $B_1^{(2)} = B_2^{(2)} = B_1^{(1)} = B_2^{(1)}$.

(b) *If* $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *then*

$$J'C + S_{11}'S_{12} = J_{(k-1)\times 2} + \begin{pmatrix} 0 \;\cdots\; 0 \\ S_1 \end{pmatrix} = \begin{pmatrix} 1 \;\cdots\; 1 \\ S_2 \end{pmatrix}$$

and $S_{11}'S_{13} = \begin{pmatrix} 0 \;\cdots\; 0 \\ S_3 \end{pmatrix}$, *which implies*

$$\lambda_{1k} = \lambda_{1(k+1)} = 1 \quad \text{and} \quad \lambda_{1j} = 0 \quad \text{for} \quad k+1 < j \leq v. \tag{A.6}$$

Since $A_1'A_1 = A_2'A_2 = (\lambda_{ij})$, *(A.4) and (A.6) imply that* $B_i^{(1)} = B_i^{(2)}$, $i = 1,2$.

Case (ii). $r_j = 3$ *for all* $1 \leq j \leq k-1$. *We may assume*

$$S_{11} = \left(S_1 \;\vdots\; S_2 \;\vdots\; S_3\right)$$

where each S_i *is a* $3\times k_i$ *matrix with one's in the* ith *row and* zero elsewhere, $k_1 > k_2 > k_3 > 0$, $\sum_{i=1}^{3} k_i = k-1$.

Then $A_1'A_1$ is of the form (A.3), but

$$J'J + S_{11}'S_{11} = 2J_{(k-1)\times(k-1)} + \left(\begin{array}{c:c:c} S_1'S_1 & S_1'S_2 & S_1'S_3 \\ \hdashline S_2'S_1 & S_2'S_2 & S_2'S_3 \\ \hdashline S_3'S_1 & S_3'S_2 & S_3'S_3 \end{array}\right)$$

$$= 2J_{(k-1)\times(k-1)} + \left(\begin{array}{c:c:c} K_{k_1\times k_1} & \underset{\sim}{0} & \underset{\sim}{0} \\ \hdashline \underset{\sim}{0} & J_{k_2\times k_2} & \underset{\sim}{0} \\ \hdashline \underset{\sim}{0} & \underset{\sim}{0} & J_{k_3\times k_3} \end{array}\right)$$

$$= \left(\begin{array}{c:c:c} 3J_{k_1\times k_1} & 2J_{k_1\times k_2} & 2J_{k_1\times k_3} \\ \hdashline 2J_{k_2\times k_1} & 3J_{k_2\times k_2} & 2J_{k_2\times k_3} \\ \hdashline 2J_{k_3\times k_1} & 2J_{k_3\times k_3} & 3J_{k_3\times k_3} \end{array}\right) \tag{A.7}$$

Set $k_{-1} = k_0 = 0$, (A.7) implies that for $t = 1,2,3$ and $k_{t-2} + k_{t-1} < i \leq k_{t-2} + k_{t-1} + k_t$

$$\lambda_{ij} = \begin{cases} 3, & k_{t-2} + k_{t-1} < j \leq k_{t-2} + k_{t-1} + k_t \\ 2, & 1 \leq j \leq k-1 \text{ but } j \notin (k_{t-2} + k_{t-1}, k_{t-2} + k_{t-1} + k_t]. \end{cases}$$

(a) If $C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, then

$$J'C + S'_{11}S_{12} = \begin{pmatrix} 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \end{pmatrix} + S'_{11}S_{12},$$

which implies $\lambda_{ik} \geq 2$, for $1 \leq i \leq k-1$.

Since $A'_1A_1 = A'_2A_2 = (\lambda_{ij})$, it forces $B_i^{(2)} = B_1^{(1)}$ for $i = 1$ or $i = 2$.

(b) If $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$J'C + S'_{11}S_{12} = J_{(k-1)\times 2} + S'_{11}S_{12},$$

which implies $\lambda_{ij} \geq 1$ for $1 \leq i \leq k-1$, $j = k, k+1$. If $k_t \neq 0$ for $t = 1,2$, then either $B_1^{(2)}$ or $B_2^{(2)}$ has to be equal to one of $B_1^{(1)}$ or $B_2^{(1)}$. If $k_2 = k_3 = 0$, also forces either $B_1^{(2)}$ or $B_2^{(2)}$ equals to one of $B_1^{(1)}$ or $B_2^{(1)}$, otherwise $B_3^{(2)}$ has at least $k + 1$ one's which is a contradiction.

By using (A.1) and Theorem A.1, it is easy to obtain:

Corollary A.2. There is no $(v,3)$ trade of volume 5.

Lemma A.4. Suppose $D_1:D_2$ is a (v,k) trade of volume 5 and there is no $(v,k-1)$ trade of volume 5. Then the number n of varieties appearing in $D_1:D_2$ is at least $2k$.

Proof: If $n \leq 2k-1$, $D_1:D_2$ can be considered as a $(2k-1,k)$ trade of volume 5.

Lemma A.2 implies $D_1^c:D_2^c$ is a $(2k-1,k-1)$ trade of volume 5 which is a contradiction.

Theorem A.2. There is no $(v,4)$ trade of volume 5.

Proof: Suppose $D_1:D_2$ is a $(v,4)$ trade of volume 5. By Theorem A.1, any two blocks in D_i has at most 2 elements in common, triplet will not occur in block intersections. To examine the existence of such trades, it is enough to consider three cases: (1) each pair appears in D_i at most once; (2) there is a pair appearing in D_i twice and other pairs appearing in D_i at most twice; (3) there is a pair appearing in D_i three times.

It is enough to start with D_1.

Case (1). This is indeed the case $|B_i^{(1)} \cap B_j^{(1)}| = 1$ for all i and j. Since if $|B_i^{(1)} \cap B_j^{(1)}| = 0$ for some i and j, the number of varieties appearing in D_1 would be greater than 10 which contradicts to (A.1). Ten varieties will be involved, each appearing twice. With no loss, we may assume

$$A_1 = \begin{pmatrix} 1&1&1&1&0&0&0&0&0&0\\ 1&0&0&0&1&1&1&0&0&0\\ 0&1&0&0&1&0&0&1&1&0\\ 0&0&1&0&0&1&0&1&0&1\\ 0&0&0&1&0&0&1&0&1&1 \end{pmatrix}$$

To satisfy the conditions $A_1'A_1 = A_2'A_2$ *and* $|B_i^{(1)} \cap B_j^{(1)}| = |B_i^{(2)} \cap B_j^{(2)}| = 1$ *for all* i *and* j, A_2 *has to be equal to* A_1.

Case (2). $\max_{i,j\in V} \lambda_{ij} = 2$. *Let* $r_1 = \max_i r_i$ *and* $r_2 = \max_j r_j$, *where* (i,j) *runs through all the pairs with* $\lambda_{ij} = 2$. *Then* $(r_1,r_2) = (2,2)$ *or* $(3,2)$ *or* $(3,3)$.

(i) *If* $r_1 = r_2 = 2$, *then*

$$A_1 = \left(\begin{array}{cc|cccccccc} 1&1&1&1&0&0&0&0&0&0\\ 1&1&0&0&1&1&0&0&0&0\\ \hline 0&0& & & & & & & & \\ 0&0& & & & S_1 & & & & \\ 0&0& & & & & & & & \end{array}\right)$$

$$A_2 = \left(\begin{array}{cc|cccccccc} 1&1&1&0&1&0&0&0&0&0\\ 1&1&0&1&0&1&0&0&0&0\\ \hline 0&0& & & & & & & & \\ 0&0& & & & S_2 & & & & \\ 0&0& & & & & & & & \end{array}\right)$$

with

$$S_2 = \begin{pmatrix} 1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1 \end{pmatrix} \qquad \textit{which is no good}$$

or $$S_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} .$$

The latter forces

$$S_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and hence $B_5^{(1)} = B_5^{(2)}$, *a contradiction*.

(ii) *If* $r_1 = 3$, *but* $r_2 = 2$, *then*

$$A_1 = \left(\begin{array}{cc|cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & & & & & & & & \\ 0 & 0 & & & & S_1 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right)$$

which implies

$$A_2 = \left(\begin{array}{cc|cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & & & & & & & & \\ 0 & 0 & & & & S_2 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right) .$$

This implies $B_3^{(1)} = B_3^{(2)}$, *a contradiction*.

(iii) If $r_1 = r_2 = 3$, $n \leq 9$ and

$$A_1 = \left(\begin{array}{cc:cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 1 & 0 & & & & & & & & \\ 0 & 1 & & & & S_1 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right)$$

which implies $\lambda_{ij} = 1$ for $i = 1,2$ and $3 \leq j \leq 6$. Also $\lambda_{34} \geq 1$ and $\lambda_{56} \geq 1$. There are four possibilities for A_2:

(a)
$$A_2 = \left(\begin{array}{cc:cccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hdashline 1 & 0 & & & & & & & & \\ 0 & 1 & & & & S_2 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right) .$$

Then

$$S_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

implies $|B_3^{(2)} \cap B_4^{(2)}| = 3$ which is impossible.

(b)
$$A_2 = \left(\begin{array}{cc:cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & & & & & & & & \\ 0 & 1 & & & & S_2 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right)$$

which implies

$$S_1 = \begin{pmatrix} x_3 & x_4 & x_5 & 0 & 1 & x_8 & x_9 & 0 \\ y_3 & y_4 & y_5 & 0 & 1 & y_8 & y_9 & 0 \\ 1 & u_1 & 1 & 1 & 0 & v_1 & w_1 & 0 \end{pmatrix}$$

and

$$S_2 = \begin{pmatrix} x_3 & x_4 & x_5 & 1 & 0 & x_8 & x_9 & 0 \\ y_3 & y_4 & y_5 & 1 & 0 & y_8 & y_9 & 0 \\ 1 & 1 & u_2 & 0 & 1 & v_2 & w_2 & 0 \end{pmatrix} .$$

S_2 *implies* $\lambda_{74} = 2$ *and hence* $x_4 = y_4 = 1$ *and* $u_1 = 0$. S_1 *implies* $\lambda_{56} = 2$ *and hence* $x_5 = y_5 = 1$, $u_2 = 0$. *But these imply that* $|B_3^{(1)} \cap B_4^{(1)}| \geq 3$ *and* $|B_3^{(2)} \cap B_4^{(2)}| \geq 3$, *a contradiction*.

(c)
$$A_2 = \left(\begin{array}{cc:cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hdashline 1 & 0 & & & & & & & & \\ 0 & 1 & & & & S_2 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right)$$

implies

$$S_1 = \left(\begin{array}{ccccccc:c} x_3 & 0 & x_5 & 0 & 1 & 1 & x_9 & 0 \\ y_3 & 0 & y_5 & 0 & 1 & 1 & y_9 & 0 \\ \hdashline & & & S_{11} & & & & 0 \end{array}\right)$$

and

$$S_2 \left(\begin{array}{ccccccc:c} x_3 & 1 & x_5 & 1 & 0 & 0 & x_9 & 0 \\ y_3 & 1 & y_5 & 1 & 0 & 0 & y_9 & 0 \\ \hdashline & & & S_{21} & & & & 0 \end{array}\right) .$$

S_2 *implies* $\lambda_{46} = 2$ *but there is no way to arrange* S_1 *with such property*.

(d)
$$A_2 = \left(\begin{array}{cc|cccccccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & & & & & & & & \\ 0 & 1 & & & & S_2 & & & & \\ 0 & 0 & & & & & & & & \end{array}\right).$$

This implies $B_3^{(1)} = B_3^{(2)}$ and $B_4^{(1)} = B_4^{(2)}$.

Case (3). $\max\limits_{i,j\in V} \lambda_{ij} = 3$. Assume $r_1 = r_2 = \lambda_{12} = 3$ (again $n \le 9$).

$$A_1 = \left(\begin{array}{cc|cccccc|cc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & & & & & & & 0 \\ 0 & 0 & & & S_1 & & & & & 0 \end{array}\right)$$

implies $\lambda_{ij} = 1$ for $i = 1,2$ and $3 \le j \le 8$. Also $\lambda_{34} \ge 1$, $\lambda_{56} \ge 1$, $\lambda_{78} \ge 1$ and $r_{10} = 0$. Therefore

$$A_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & y & z & u & 1 & 1 & 0 & 0 \end{pmatrix}$$

with two of x,y,z,u equal to one. But there is no way to arrange S_1.

All the three cases are ruled out, we now have the desired result.

REFERENCES

1. Foody, W., (1980). Properties, construction, and application of BIB designs with repeated blocks. Ph.D. Thesis. University of Illinois at Chicago Circle.

2. Foody, W. and Hedayat, A., (1977). On theory and applications of BIB designs with repeated blocks. Ann. Statist. 5, 932-945.

3. Graham, R.L., Li, S.-Y.R. and Li, W.-C.W., (1980). On the structure of t-designs. SIAM J. of Algebraic and Discrete Methods, 1, 8-14.

4. Graver, J.E. and Jurkat, W.B., (1973). The module structure of integral designs. J. Comb. Theory. (A) 15.

5. Hedayat, A. and John, P.W.M., (1974). Resistant and susceptible BIB designs. Ann. Statist. 2, 148-158.

6. Hedayat, A. and Khosrovshahi, G.B. (1979). An algebraic study of BIB designs: A complete solution for $v = 6$ and $k = 3$. J. Comb. Theory. (A), to appear.

7. Hedayat, A. and Li, S.-Y.R., (1979). The trade off method in the construction of BIB designs with variable support size. Ann. Statist. 7, 1277-1287.

8. Mann, H.B., (1969). A note on balanced incomplete block designs. Ann. Math. Statist. 40, 679-680.

9. Parker, E.T., (1963). Remarks on balanced incomplete block designs. Proc. Amer. Math. Soc. 14, 729-730.

10. Seiden, E., (1963). A supplement to Parker's "Remarks on balanced incomplete block designs". Proc. Amer. Math. Soc. 14, 731-732.

11. Stanton, R.G. and Sprott, D.A., (1964). Block intersections in balanced incomplete block designs. Canad. Math. Bull. 7, 539-548.

12. Stanton, R.G., (1970). Some results on BIBD's with repeated blocks. Sankhya Series A, 457-458.

13. van Lint, J.H., (1973). Block designs with repeated blocks and $(b,r,\lambda) = 1$. J. Comb. Theory 15, 289-309.

14. van Lint, J.H., (1974). Combinatorial Theory Seminar, Eindhoven University of Technology: Lecture Notes in Mathematics. No. 382 (New York: Springer-Verlag).

REFERENCES

15. van Lint, J.H. and Ryser, H.J., (1972). Block designs with repeated blocks. _Discrete Math._ _3_, 381-396.

16. Wynn, H.P., (1977). Convex sets of finite population plans. _Ann. Statist._ _5_, 414-418.